Eur. Phys. J. C 20, 147–159 (2001)
Digital Object Identifier (DOI) 10.1007/s100520100639

THE EUROPEAN PHYSICAL JOURNAL C

Derivative expansion for the effective action of chiral gauge fermions. The normal parity component

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Received: 31 January 2001 / Published online: 12 April 2001 – © Springer-Verlag 2001

Abstract. Explicit exact formulas are presented, up to fourth order in a strict chiral covariant derivative expansion, for the normal parity component of the Euclidean effective action of even-dimensional Dirac fermions. The bosonic background fields considered are scalar, pseudo-scalar, vector and axial vector. No assumptions are made on the internal symmetry group and, in particular, the scalar and pseudo-scalar fields need not be on the chiral circle.

1 Introduction

The effective action plays an important role in quantum field theory, both from the phenomenological and from the formal point of view, since it compactly embodies the renormalized properties of the system; Green's functions, S-matrix elements and the expectation values of observables can be extracted from it. In addition it is suitable to study non-perturbative issues. Consequently the effective action functional has been the subject of a very large amount of work from different points of view, which include its proper definition and renormalization, its symmetries and their anomalous breaking, its properties in curved space-times or finite temperature, and its calculation. An excellent and quite complete review for chiral fermions in flat space-time and zero temperature, the case of interest to us, can be found in [1]. For anomalies in curved space-times see e.g. [2]. Work on chiral fermions at finite temperature can be found in [3,4].

In the present work and its companion paper [5] we concentrate on the computational issues. Concretely we deal with the calculation of the effective action of Dirac fermions in the presence of bosonic external fields; thus, the functional adds one-loop Feynman diagrams with fermions running on the loop and bosonic external legs. We restrict ourselves to the case of even dimensions (since there are some technical differences with the odd-dimensional case), zero temperature and flat space-time. The class of external bosonic fields to be included is that of scalar, pseudo-scalar, vector and axial vector fields. Coupling to higher tensor fields is not included; nevertheless, the class considered here is quite large since no assumption will be made on the internal symmetry group; that is, the external fields are arbitrary matrices free from algebraic assumptions regarding their dimension, commutativity, chiral circle constraint or any other constraints.

The computation of the effective action functional in closed form is not possible in general, and thus several asymptotic expansions have been devised. In the heat kernel expansion the terms are classified by its scale dimension; that is, each term has a well-defined number of external fields and derivative operators. It is the computationally simplest expansion and so it has been carried out to considerably large orders even for curved space-times [6]. Other expansions can be regarded as resummations of this one. In the perturbative expansion, the contributions are classified by the number of external legs, that is, the number of external fields in the term, and all orders in their momenta are added. This is a weak field approximation which however captures the non-locality of the exact functional. On the other hand, in the covariant derivative expansion, to be considered in this work, the contributions are classified by the number of chiral covariant derivatives, or equivalently, by the number of Lorentz indices. Both definitions are equivalent in the absence of external tensor fields. This counting is appropriate for external fields with a smooth space-time dependence and weak gauge fields. The scalar and pseudo-scalar fields need not be weak, and this allows one to study non-perturbative issues such as spontaneous symmetry breaking. Two important properties of this expansion is that chiral symmetry is preserved (modulo anomalies) separately for each term, and that the terms are local.

The effective action is a ultraviolet divergent quantity which needs to be renormalized. The non-perturbative definition of this functional in the general chiral case is not so straightforward as for vector-like theories (in which there are no pseudo-scalar nor axial fields) due to the presence of essential chiral anomalies which affect the imaginary part (the phase of the fermionic determinant) [7]. Nevertheless such a definition exists [1] and so the effective action is perfectly well defined also in the chiral case, displaying only the standard renormalization ambiguities in the form of polynomial terms. The subtleties in the non-perturbative definition of the effective action are much alleviated within the asymptotic expansions noted above.

In particular, within the covariant derivative expansion to be worked out in this paper for the real part of the effective action and in [5] for the imaginary part, there is only a finite number of ultraviolet divergent terms. They are afflicted by polynomial ambiguities (including anomalies and multivaluation) but are otherwise unique. All higher order terms are ultraviolet finite and so they are unambiguous.

It should be emphasized that the covariant derivative expansion of the effective action functional, although asymptotic, is perfectly well defined; that is, it does not depend on how it is written or computed. (This is not true for other expansions such as the commutator expansion, as we will discuss below.) This is because it corresponds to classify the terms by their scaling under (covariant) dilatations of the external fields. Each of these terms is a well-defined and universal functional, in the sense that they hold for any possible internal symmetry group, and the same is true for the perturbative or heat kernel expansions. For these two latter expansions it is relatively easy to write an explicit form for each term without putting restrictions on the internal symmetry group. The reason for this is that all non-commutative quantities, namely the matrix-valued external fields, are treated perturbatively and each term contains only a finite number of them. On the other hand, in a strict covariant derivative expansion the gauge fields are treated perturbatively but the scalar and pseudo-scalar fields appear to all orders in every single term, and these two fields do not commute with each other. Nevertheless, in this work and in [5] we show that the terms of the derivative expansions are also fully amenable to explicit computation in closed form without assuming particular properties of the internal symmetry group. In addition, no chiral rotation (or diagonalization) is required to express these universal functionals. Our result takes an analytical form in terms of the external fields.

In this work we deal with the real part of the effective action and compute it until fourth order in the covariant derivative expansion for arbitrary even space-time dimensions. The real part is the simplest one since it is free from essential chiral anomalies and multivaluation. It contains only a scale anomaly. In [5] the study of the imaginary part is dealt with at leading order in the derivative expansion for two- and four-dimensional space-times. Besides the results themselves, in this paper and in [5] we introduce notational conventions which are very well suited to the chiral problem. In particular, we find that the formulas for the real part in the full chiral case are identical to those of the vector-like case.

In Sect. 2 we introduce our notational conventions, some of which are not standard, and also introduce the effective action. In Sect. 3 we show how, for the real part of the effective action, an appropriate notation allows to reduce the full chiral case to the vector-like case, and also to carry out explicitly all integrations over the momentum in the fermionic loop. The section ends with explicit formulas for the real part of the effective action up to four covariant derivatives, based in the convenient method introduced by Chan for bosons [8]. Finally, in Sect. 4 we illustrate the meaning of the formulas by analyzing the case of second order and two space-time dimensions and a particular case is worked out explicitly. Next we show how our notation allows to obtain commutator expansions quite efficiently, and the section is ended by giving the analogous explicit formulas for bosons, once again without restriction on the internal symmetry group.

2 General considerations

2.1 The Dirac operator

The Euclidean effective action of fermions in a d-dimensional flat space (d even) is $\int \mathrm{d}^d x \bar{\psi} \mathrm{D} \psi$, where D is the Dirac operator. The class of operators to be considered is, in terms of the left–right (LR) fields,

$$\mathrm{D} = \not{D}_{\mathrm{R}} P_{\mathrm{R}} + \not{D}_{\mathrm{L}} P_{\mathrm{L}} + m_{\mathrm{LR}} P_{\mathrm{R}} + m_{\mathrm{RL}} P_{\mathrm{L}}, \tag{1}$$

where $P_{\mathrm{R,L}} = \frac{1}{2}(1 \pm \gamma_5)$ are the projectors on the subspaces $\gamma_5 = \pm 1$. Our conventions are

$$\begin{aligned} &\gamma_\mu = \gamma_\mu^\dagger, \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \\ &\gamma_5 = \gamma_5^\dagger = \gamma_5^{-1} = \eta_d \gamma_0 \cdots \gamma_{d-1}, \quad \mathrm{tr}_{\mathrm{Dirac}}(1) = 2^{d/2}. \end{aligned} \tag{2}$$

Here $\eta_d = \pm \mathrm{i}^{d/2}$ (a concrete choice will not be needed). $D_\mu^{\mathrm{R,L}} = \partial_\mu + v_\mu^{\mathrm{R,L}}$ are the chiral covariant derivatives. The external bosonic fields $v_\mu^{\mathrm{R,L}}(x)$ and $m_{\mathrm{LR}}(x)$, $m_{\mathrm{RL}}(x)$ are matrices in internal space (referred to as flavor), the identity in Dirac space and multiplicative operators in x space. Unitarity of the theory imposes restrictions on the hermiticity properties of these fields, namely, $v_{\mathrm{R,L}}$ must be antihermitian and $m_{\mathrm{LR}}^\dagger = m_{\mathrm{RL}}$. In practice these restrictions will play almost no role in the calculation[1] and m_{LR} and m_{RL} will be treated as independent variables. To avoid infrared divergences we will assume that m_{LR} and m_{RL} are nowhere singular matrices.

In terms of the vector-axial (VA) variables

$$\mathrm{D} = \not{D}_V + \not{A}\gamma_5 + S + \gamma_5 P, \tag{3}$$

where $D_\mu^V = \partial_\mu + V_\mu$ is the vector covariant derivative and

$$v_{\mathrm{R,L}} = V \pm A, \quad m_{\mathrm{LR}} = S + P, \quad m_{\mathrm{RL}} = S - P. \tag{4}$$

2.2 Chiral transformations

Chiral transformations act as follows:

$$\begin{aligned} &D_\mu^R \to \Omega_R^{-1} D_\mu^R \Omega_R, \quad D_\mu^L \to \Omega_{\mathrm{L}}^{-1} D_\mu^L \Omega_{\mathrm{L}}, \\ &m_{\mathrm{LR}} \to \Omega_{\mathrm{L}}^{-1} m_{\mathrm{LR}} \Omega_{\mathrm{R}}, \quad m_{\mathrm{RL}} \to \Omega_{\mathrm{R}}^{-1} m_{\mathrm{RL}} \Omega_{\mathrm{L}}, \end{aligned} \tag{5}$$

[1] However, we will exploit the fact that the spectrum of $m_{\mathrm{R}}^2(x)$ (defined below in (6)) is non-negative to choose the branch cuts of the logarithm or squared root functions along the negative real axis.

where $\Omega_{\rm R,L}(x)$ are independent, nowhere singular and otherwise arbitrary matrices in flavor space. Vector gauge transformations correspond to chiral transformations in the diagonal subgroup $\Omega_{\rm R}=\Omega_{\rm L}$, i.e. $\mathrm{D}\to\Omega^{-1}\mathrm{D}\Omega$.

It will be convenient to introduce the two combinations

$$m_{\rm R}^2=m_{\rm RL}m_{\rm LR},\quad m_{\rm L}^2=m_{\rm LR}m_{\rm RL},\tag{6}$$

which transform solely under $\Omega_{\rm R}$ or $\Omega_{\rm L}$, respectively. Note that the two matrices $m^2_{\rm R,L}(x)$ are related by a similarity transformations, and thus they have the same spectrum.

The different pieces in the Dirac operator transform in a well-defined manner under chiral transformations, namely, the quantities $m_{\rm LR}$, $m_{\rm RL}$, $D_{\rm R}$ and $D_{\rm L}$ fall in the chiral representations LR, RL, RR and LL, respectively; cf. (5). New objects with well-defined chirality are obtained by multiplication in the natural way, i.e., if X_{ab} falls in the representation ab and Y_{bc} in bc, for $a,b,c=\mathrm{R},\mathrm{L}$, the product $Z_{ac}=X_{ab}Y_{bc}$ (no sum over b is implied) falls in the representation ac. If in addition X_{ab} is a multiplicative operator, its chiral covariant derivative

$$(\hat D_\mu X)_{ab}=D^a_\mu X_{ab}-X_{ab}D^b_\mu,\quad a,b=\mathrm{R},\mathrm{L},\tag{7}$$

is also multiplicative. In particular,

$$\begin{aligned}\hat D_\mu m_{\rm LR}&:=(\hat D_\mu m)_{\rm LR}=\partial_\mu m_{\rm LR}+v^{\rm L}_\mu m_{\rm LR}-m_{\rm LR}v^{\rm R}_\mu,\\ \hat D_\mu m_{\rm RL}&:=(\hat D_\mu m)_{\rm RL}=\partial_\mu m_{\rm RL}+v^{\rm R}_\mu m_{\rm RL}-m_{\rm RL}v^{\rm L}_\mu.\end{aligned}\tag{8}$$

In addition,

$$F^{\rm R}_{\mu\nu}=[D^{\rm R}_\mu,D^{\rm R}_\nu],\quad F^{\rm L}_{\mu\nu}=[D^{\rm L}_\mu,D^{\rm L}_\nu],\tag{9}$$

are also chiral covariant and multiplicative. All multiplicative chiral covariant local objects come as combinations of m, F and their chiral covariant derivatives.

2.3 The effective action

The fermionic effective action $W=-\log\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left(-\int\mathrm{d}^dx\bar\psi\mathrm{D}\psi\right)$ is given by

$$W[v,m]=-\mathrm{Tr}\log(\mathrm{D}).\tag{10}$$

As is well known this expression is formal due to the presence of ultraviolet divergences. Mathematically proper definitions of $\mathrm{Tr}\log(\mathrm{D})$ exist in the literature. Here it is only necessary to emphasize that different definitions of W may at most differ by terms which are local polynomial of dimension d, that is, polynomials in the external fields v and m and their derivatives. This is a standard result of perturbative quantum field theory that can be established by isolating the ultraviolet divergent one-loop Feynman graphs, which may contain at most d insertions, and expanding on the external momenta to extract the divergent part. In practice this means that any method consistent with the formal expression can be used to make W finite, since it will give the same ultraviolet finite contributions as any other method. The effective action of a concrete physical system described by the Dirac operator D will be given by any of the renormalized versions of W plus an appropriate local polynomial counterterm.

As usual, it will be convenient to introduce the pseudo-parity transformation, $\mathrm{R}\leftrightarrow\mathrm{L}$ (that is, $v_{\rm L}\leftrightarrow v_{\rm R}$ and $m_{\rm LR}\leftrightarrow m_{\rm RL}$ or equivalently $A_\mu\to-A_\mu$, $P\to-P$) and split the effective action into its even and odd components under this transformation, $W^\pm$, i.e.,

$$W[v,m]=W^+[v,m]+W^-[v,m].\tag{11}$$

(These are also known as normal and abnormal parity components.) Due to parity invariance, which involves an additional $(x_0,\boldsymbol{x})\to(x_0,-\boldsymbol{x})$ in the fields, the pseudo-parity odd component of the effective action $W^-[v,m]$ is that containing the Levi-Civita pseudo-tensor. In addition, the $W^\pm[v,m]$ coincide with the real and imaginary parts, respectively, of the (Euclidean) effective action when the standard hermiticity for the fields is assumed.

By computing the effective action within a derivative expansion, we mean to express it as a sum of terms with a well-defined number of covariant derivative operators but any number of scalars. This is equivalent to the classification of the terms by the number of Lorentz indices they contain. We emphasize this since sometimes the expression "derivative expansion" is used in the literature to denote large mass expansions or expansions in the number of all kind of commutators; that is, those implied by the covariant derivative plus those of the form $[m^2,\ ]$, etc. Further, we mean to write each of the terms using building blocks which are multiplicative operators in x space (i.e., functions of x and not differential operators) and which are trivial in Dirac space (since the Dirac trace can be explicitly computed for any given order in a derivative expansion). The trace in flavor space will not be worked out since we are not imposing any algebraic constraint on the flavor structure of the fields.

Note that, because d is even and all Lorentz invariants are formed with $\delta_{\mu\nu}$ and $\epsilon_{01\ldots d-1}$, there are no odd-order terms in the derivative expansion.

In this work we will consider the pseudo-parity even component of the effective action which is simpler due to its lack of chiral anomalies. The pseudo-parity odd component is worked out in [5].

2.4 Local basis in flavor space

Occasionally it will be useful to diagonalize the flavor matrices $m_{\rm LR}(x)$ and $m_{\rm RL}(x)$. This can be done by solving the eigenvalue problem (at each point x)

$$\begin{pmatrix}0 & m_{\rm RL}\\ m_{\rm LR} & 0\end{pmatrix}\begin{pmatrix}|j,\mathrm{R}\rangle\\ \pm|j,\mathrm{L}\rangle\end{pmatrix}=\pm m_j\begin{pmatrix}|j,\mathrm{R}\rangle\\ \pm|j,\mathrm{L}\rangle\end{pmatrix}\tag{12}$$

which yields

$$\begin{aligned}m_{\rm LR}|j,\mathrm{R}\rangle&=m_j|j,\mathrm{L}\rangle,\quad m_{\rm RL}|j,\mathrm{L}\rangle=m_j|j,\mathrm{R}\rangle,\\ \langle j,\mathrm{L}|m_{\rm LR}&=m_j\langle j,\mathrm{R}|,\quad \langle j,\mathrm{R}|m_{\rm RL}=m_j\langle j,\mathrm{L}|.\end{aligned}\tag{13}$$

$\langle j,\mathrm{R}|$ is the dual basis of $|j,\mathrm{R}\rangle$, $\langle j,\mathrm{R}|k,\mathrm{R}\rangle=\delta_{jk}$ (no orthonormality of the basis is implied). The numbers m_j^2 are the common eigenvalues of m_{R}^2 and m_{L}^2 and $|j,\mathrm{R}\rangle$ and $|j,\mathrm{L}\rangle$ are their eigenvectors. Because $m_{\mathrm{LR}}^\dagger=m_{\mathrm{RL}}$, the eigenvalues m_j can be taken to be positive.

2.5 Some notational conventions

In this section we will introduce some notational conventions which are essential for carrying out the subsequent calculations. They are also used in [5].

Because W^+ is by definition invariant under the exchange of the labels R and L, each term T in the expansion of W^+ will have a pseudo-parity conjugate term, T^*, obtained from T by exchanging everywhere the labels R and L. (Note that due to the cyclic property, it may actually happen that T and T^* coincide.) Thus we will use the following convention:

Convention 1. In W^+, the terms T and T^* will be identified, so that under this Convention T actually stands for $(1/2)(T+T^*)$.

It is important to note that the word "term" is used here in a very specific way, namely, it refers only to contributions of the form $T=\mathrm{tr}(X)$, that when added produce the effective action. (By extension, term may denote also the quantity X itself. In what follows two such quantities X and Y differing only by a cyclic permutation will be identified, since they are equivalent inside the trace.) Of course, the identification implied in Convention 1 does not apply to smaller pieces or factors inside each term.

Due to chiral covariance each term is a product of factors with well-defined chirality (namely, LR, RL, RR or LL) correctly combined to preserve chirality (i.e., $\cdots X_{ab}Y_{bc}\cdots$). In addition, due to the cyclic property of the trace, if a term starts with label $a=\mathrm{R},\mathrm{L}$ it has to end also with same label a, e.g. $\mathrm{tr}(X_{\mathrm{RR}}Y_{\mathrm{RL}}Z_{\mathrm{LR}})$. Based on these observations we will make the following convention:

Convention 2. In expressions where the chiral labels are combined preserving chirality, these labels are redundant and will be suppressed; thus, a term such as $\mathrm{tr}(X_{\mathrm{RR}}Y_{\mathrm{RL}}Z_{\mathrm{LR}})$ will be written as $\mathrm{tr}(\mathsf{XYZ})$. Note that $\mathrm{tr}(\mathsf{XYZ})$ could be expanded either as $\mathrm{tr}(X_{\mathrm{RR}}Y_{\mathrm{RL}}Z_{\mathrm{LR}})$ or $\mathrm{tr}(X_{\mathrm{LL}}Y_{\mathrm{LR}}Z_{\mathrm{RL}})$, but both expressions are equivalent under Convention 1. We will choose the first label as R.

For instance,

$$\begin{aligned}\mathrm{tr}(\mathsf{F}_{\mu\nu}\hat{\mathsf{D}}_\mu\mathsf{m}\hat{\mathsf{D}}_\nu\mathsf{m}) &= \mathrm{tr}(F^{\mathrm{R}}_{\mu\nu}\hat{D}_\mu m_{\mathrm{RL}}\hat{D}_\nu m_{\mathrm{LR}})\\ &= \frac{1}{2}\mathrm{tr}(F^{\mathrm{R}}_{\mu\nu}\hat{D}_\mu m_{\mathrm{RL}}\hat{D}_\nu m_{\mathrm{LR}})\\ &\quad+\frac{1}{2}\mathrm{tr}(F^{\mathrm{L}}_{\mu\nu}\hat{D}_\mu m_{\mathrm{LR}}\hat{D}_\nu m_{\mathrm{RL}}).\end{aligned}\tag{14}$$

This notation allows one to use the objects m, v_μ, etc, as ordinary operators (i.e., elements of an associative algebra). For instance, the property $\mathsf{m}f(\mathsf{m})=f(\mathsf{m})\mathsf{m}$ is verified as is readily checked. With this notation (8) and (9) become

$$\hat{\mathsf{D}}_\mu\mathsf{m}=\mathsf{D}_\mu\mathsf{m}-\mathsf{m}\mathsf{D}_\mu=[\mathsf{D}_\mu,\mathsf{m}],\quad \mathsf{F}_{\mu\nu}=[\mathsf{D}_\mu,\mathsf{D}_\nu],\tag{15}$$

and (13) becomes

$$\mathsf{m}|\mathsf{j}\rangle=m_j|\mathsf{j}\rangle,\quad \langle\mathsf{j}|\mathsf{m}=m_j\langle\mathsf{j}|.\tag{16}$$

Another essential property, the cyclic property of the trace, holds too. For instance,

$$\begin{aligned}\mathrm{tr}(\mathsf{X}\mathsf{m}) &= \mathrm{tr}(X_{\mathrm{RL}}m_{\mathrm{LR}})=\mathrm{tr}(m_{\mathrm{LR}}X_{\mathrm{RL}})\\ &= \mathrm{tr}(m_{\mathrm{RL}}X_{\mathrm{LR}})=\mathrm{tr}(\mathsf{m}\mathsf{X}).\end{aligned}\tag{17}$$

Let us note that these conventions have to be slightly modified to include the pseudo-parity odd component $W^-[m,v]$. In particular, the cyclic property of the trace is modified and this fact is at the origin of the chiral anomaly in this formalism [5].

Below, it will be necessary to carry out parametric integrations where the parameters appear in different places tied to operators which do not commute. To do these integrations another convention is needed regarding the order of operators. Consider for instance the integral

$$\int\frac{\mathrm{d}^dp}{(2\pi)^d}\mathrm{tr}\left[\frac{\mathsf{m}}{p^2+\mathsf{m}^2}\mathsf{v}_\mu\frac{\mathsf{m}}{p^2+\mathsf{m}^2}\mathsf{v}_\mu\right].\tag{18}$$

It can be computed by first taking the trace using the basis of eigenvectors of m introduced in Sect. 2.4. This yields an expression involving matrix elements of v_μ in this basis, namely,

$$\int\frac{\mathrm{d}^dp}{(2\pi)^d}\sum_{j,k}\frac{m_j}{p^2+m_j^2}(\mathsf{v}_\mu)_{jk}\frac{m_k}{p^2+m_k^2}(\mathsf{v}_\mu)_{kj},\tag{19}$$

where $(\mathsf{v}_\mu)_{jk}=\langle j,\mathrm{L}|v_\mu^{\mathrm{L}}|k,\mathrm{L}\rangle$ and $(\mathsf{v}_\mu)_{kj}=\langle k,\mathrm{R}|v_\mu^{\mathrm{R}}|j,\mathrm{R}\rangle$. (For definiteness we have assumed that the operator inside the trace in (18) has been expanded as a RR term.) The point of taking matrix elements is that, since now everything is in terms of commuting numbers, the momentum integral can be done explicitly. Instead of that, we will use an equivalent but preferable procedure, namely, we will label the matrices m according to their position relative to v_μ. Since there are two v_μ there are three possible relative positions which are labeled by 1, 2 and 3. With this prescription, the same integral can be represented unambiguously by the formula

$$\int\frac{\mathrm{d}^dp}{(2\pi)^d}\mathrm{tr}\left[\frac{\mathsf{m}_1}{p^2+\mathsf{m}_1^2}\frac{\mathsf{m}_2}{p^2+\mathsf{m}_2^2}\mathsf{v}_\mu^2\right].\tag{20}$$

The usefulness of this notation is that, since the ordering is given by the labels, m_1 and m_2 are effectively c-numbers and the momentum integration can be carried out straightforwardly.

Convention 3. In an expression $f(A_1,B_2,\dots)XY\cdots$ the ordering labels $1,2,\dots$ will denote the actual position of the operators $A,B,\dots$ relative to the fixed elements $X,Y,\dots$ so that A is to be placed before X, B between X and Y, etc. That is, for a separable function $f(a,b,\dots)=\alpha(a)\beta(b)\cdots$, the expression stands for $\alpha(A)X\beta(B)Y\cdots$

Note that the fixed elements $X,Y,\dots$ appear only as simple factors (i.e. perturbatively) whereas the labeled

operators $A, B, \ldots$ may appear with a non-perturbative functional dependence. Also, note that several different labeled operators can have the same label provided that they commute[2]. Finally, in an expression with n fixed elements inside the trace, the labels 1 and $n+1$ are equivalent, due to the cyclic property of the trace. So, in the previous example, $\mathsf{m}_3 = \mathsf{m}_1$. We remark that Convention 3 is independent of Conventions 1 and 2.

The usefulness of this notation can be further exposed through the following observation regarding commutators. Let X be a single fixed element and let the symbol D_A denote the operation $[A, \ ]$, thus it immediately follows from Convention 3 that $D_A = A_1 - A_2$; that is,

$$f(D_A)X = f(A_1 - A_2)X. \tag{21}$$

For instance,

$$\mathrm{e}^{A}X\mathrm{e}^{-A} = \mathrm{e}^{A_1}\mathrm{e}^{-A_2}X = \mathrm{e}^{A_1-A_2}X = \mathrm{e}^{D_A}X, \tag{22}$$

which is a well-known identity. (Alternatively, this example can be regarded as a proof of (21).) This observation will prove useful to carry out commutator expansions (cf. Section 4.2).

As another illustration of Convention 3, consider the following identity, where $f(A)$ depends on the operator A and δA represent some first order variation of it:

$$\delta f(A) = \frac{f(A_1) - f(A_2)}{A_1 - A_2}\delta A. \tag{23}$$

Here δA is the fixed element referred to in the Convention and A_1 and A_2 refer to A before and after δA, respectively. This identity can be proven as follows[3]. As is well known

$$\delta \mathrm{e}^{sA} = \int_0^s \mathrm{d}t \mathrm{e}^{tA}\delta A \mathrm{e}^{(s-t)A}, \tag{24}$$

where s is a c-number parameter. Using Convention 3, this can be rewritten as

$$\delta \mathrm{e}^{sA} = \int_0^s \mathrm{d}t \mathrm{e}^{tA_1+(s-t)A_2}\delta A = \frac{\mathrm{e}^{sA_1} - \mathrm{e}^{sA_2}}{A_1 - A_2}\delta A, \tag{25}$$

where in the last step we have used that A_1 and A_2 are commuting quantities. This identity is then generalized for arbitrary $f(A)$ by Fourier transformation. From another point of view, noting that $A_1 - A_2$ is equivalent to $[A, \ ]$ for any A, (23) is equivalent to $[A, \delta f(A)] = [f(A), \delta A]$, which is a trivial consequence of $\delta[A, f(A)] = 0$.

Inside a trace the two ordering labels 1 and 2 become identical due to the cyclic property; therefore, (23) yields

$$\mathrm{Tr}\,(\delta f(A)) = \mathrm{Tr}\,(f'(A)\delta A)\,. \tag{26}$$

(f' denotes the derivative of f.) The corresponding formula for two successive first order variations, is

$$\mathrm{Tr}\,(\delta'\delta f(A)) = \mathrm{Tr}\left(f'(A)\delta'\delta A + \frac{f'(A_1) - f'(A_2)}{A_1 - A_2}\delta' A \delta A\right), \tag{27}$$

which in particular implies $\mathrm{Tr}\,([\delta', \delta]f(A)) = \mathrm{Tr}(f'(A)[\delta', \delta]A)$.

3 Calculation of W^+ to fourth order

3.1 Reduction to a vector-like theory

A first benefit of Conventions 1 and 2 is that the functional $W^+[v, m]$ will be formally identical to the effective action of a vector-like theory. That is, if $W_{\mathrm{V}}[V, S]$ denotes the effective action functional when $A_\mu = P = 0$,

$$W^+[v, m] = W_{\mathrm{V}}[\mathsf{v}, \mathsf{m}], \tag{28}$$

provided that $W^+[v, m]$ and $W_{\mathrm{V}}[V, S]$ have been renormalized preserving chiral and vector gauge invariances, respectively. This is because when the functional W^+, expressed in our notation in terms of v and m, is applied to the vector-like case (i.e., $v_{\mathrm{R}} = v_{\mathrm{L}}$, $m_{\mathrm{LR}} = m_{\mathrm{RL}}$) no simplification occurs; it remains unchanged in our notation. (W^- vanishes for vector-like configurations.) As a consequence, W^+ for the general case can be obtained by computing only the effective action for vector-like configurations.

It is an essential point of this discussion that we are considering field configurations which are free of constraints in flavor space: the calculation of $W^+[v, m]$ through $W_{\mathrm{V}}[V, S]$ requires V_μ and S to be generic in flavor space for the formal replacements $V \to \mathsf{v}$ and $S \to \mathsf{m}$ in the formulas to be well defined. For instance, if the calculation of $W_{\mathrm{V}}[V, S]$ were carried out in the particular case of commuting V_μ and S, $[V_\mu, S]$ would be identified with zero and this would result in an ambiguity in $W^+[v, m]$ by terms $[\mathsf{v}_\mu, \mathsf{m}]$; however, $[\mathsf{v}_\mu, \mathsf{m}]_{\mathrm{RL}} = [V_\mu, m_{\mathrm{RL}}] + \{A_\mu, m_{\mathrm{RL}}\}$, which does not vanish even for Abelian flavor groups, due to the axial term. Therefore the proper procedure is first to compute $W_{\mathrm{V}}[V, S]$ in general, then make the replacements $V \to \mathsf{v}$ and $S \to \mathsf{m}$, and afterwards apply the resulting formulas to the case at hand. This replacement is a kind of analytical continuation from the vector-like to the chiral case.

3.2 Chan's method

Here we present the most efficient method to compute W^+. Further heuristic considerations are made in Sect. 4. We have already reduced the problem to a vector-like one, with effective operator $\mathrm{D} = \not{\mathsf{D}} + \mathsf{m}$. Next, the theory is reduced to a bosonic one using $\mathrm{D}^\dagger\mathrm{D}$ as operator. This procedure is standard. We have

$$\mathrm{D} = \not{\mathsf{D}} + \mathsf{m}, \quad \mathrm{D}^\dagger = \gamma_5\mathrm{D}\gamma_5 = -\not{\mathsf{D}} + \mathsf{m}. \tag{29}$$

[2] The separation into fixed elements and labeled operators is a matter of convenience. It is also possible to label all operators to indicate their relative position and this is often useful in order to carry out algebraic manipulations with a computer.

[3] Alternatively it can be proven by starting from $\delta(A^{-1}) = -A^{-1}\delta A A^{-1} = -A_1^{-1}A_2^{-1}\delta A$ and applying it to $f(A) = (2\pi\mathrm{i})^{-1}\int_\Gamma \mathrm{d}z f(z)/(z - A)$, where Γ is positively oriented and encloses the spectrum of A, or also it can be established by considering functions of the form $f(x) = x^n$.

The second equality implies that D and $\mathrm{D}^\dagger$ are related by a similarity transformation, and so they can be assigned the same determinant, thus:

$$W^+[v,m] = -\frac{1}{2}\mathrm{Tr}\log\left(\mathrm{D}^\dagger\mathrm{D}\right). \tag{30}$$

Two remarks are in order. First, the notation $\mathrm{D}^\dagger$ assumes that the fields have the standard hermiticity, but only the fact that $\mathrm{D}^\dagger$ is similar to D is essential. Second, within a given particular renormalization prescription (e.g. the ζ-function) (30) will be correct modulo local polynomial counterterms. This is entirely sufficient since, as noted above, the action only determines the effective action modulo a local polynomial of dimension d.

The bosonic operator $\mathrm{D}^\dagger\mathrm{D}$ can be worked out to give

$$\begin{aligned}\mathrm{D}^\dagger\mathrm{D} &= -\mathsf{D}_\mu^2 - \frac{1}{2}\sigma_{\mu\nu}\mathsf{F}_{\mu\nu} - [\not{\mathsf{D}},\mathsf{m}] + \mathsf{m}^2\\ &= P^2 + U,\end{aligned} \tag{31}$$

where $\gamma_\mu\gamma_\nu = \delta_{\mu\nu} + \sigma_{\mu\nu}$, and

$$P_\mu = \mathsf{D}_\mu, \quad P^2 = -P_\mu^2, \quad U = \mathsf{m}^2 - \gamma_\mu\hat{\mathsf{D}}_\mu\mathsf{m} - \frac{1}{2}\sigma_{\mu\nu}\mathsf{F}_{\mu\nu}. \tag{32}$$

(Note that P^2 is unrelated to the pseudo-scalar field $P = (m_{\mathrm{LR}} - m_{\mathrm{RL}})/2$.)

A manifestly gauge invariant form for the derivative expansion of the effective action of a general bosonic theory with Klein–Gordon operator of the form $P^2 + U$ has been given in [8] to fourth order and in [9] to sixth order. Those formulas assume only that P_μ is a covariant derivative, i.e., that they are of the form $\partial_\mu + W_\mu(x)$, and that $U(x)$ is a Lorentz-scalar field. The fields $W_\mu(x)$ and $U(x)$ are matrices in some internal space. To fourth order the result is[4]

$$\begin{aligned}&\mathrm{Tr}\log(P^2+U)\\ &= \int\frac{\mathrm{d}^dx\mathrm{d}^dp}{(2\pi)^d}\mathrm{tr}\Bigg[-\log(N) + \frac{p^2}{d}N_\mu^2 - \frac{2p^4}{d(d+2)}\\ &\quad\times\Big(-2N_\mu^4 + (N_\mu N_\nu)^2 + 2(NN_{\mu\mu})^2\\ &\quad+4NF_{\mu\nu}NN_\mu N_\nu + (F_{\mu\nu}N^2)^2\Big) + \cdots\Bigg],\end{aligned} \tag{33}$$

where the following quantities have been defined:

$$\begin{aligned}&N = \frac{1}{p^2+U}, \quad N_\mu = [P_\mu, N], \quad N_{\mu\mu} = [P_\mu,[P_\mu,N]],\\ &F_{\mu\nu} = [P_\mu, P_\nu].\end{aligned} \tag{34}$$

In addition, N_μ^4 stands for $(N_\mu^2)^2$, and $(N_\mu N_\nu)^2 = N_\mu N_\nu N_\mu N_\nu$, etc. The symbol tr includes all internal degrees of freedom. The dots refer to terms with six or more P_μ. All terms are multiplicative operators (i.e. just functions of x) and manifestly gauge invariant. Because all terms are multiplicative, the trace cyclic property applies without restrictions. In what follows, terms related by a cyclic permutation will be identified when they appear inside the trace.

Equation (33) can straightforwardly be applied to compute W^+ with the identifications

$$N_\mu = \hat{\mathsf{D}}_\mu N, \quad N_{\mu\mu} = \hat{\mathsf{D}}_\mu^2 N, \quad F_{\mu\nu} = \mathsf{F}_{\mu\nu}, \tag{35}$$

and N and U given in (32). It is just necessary to note that in the bosonic case U is assumed to be of zeroth order; however, in the fermionic case U contains terms of first and second order in D_μ, so N has to be reexpanded in order to have a well-defined derivative expansion of W^+:

$$\begin{aligned}N^{-1} &= \Delta + U^{(1)} + U^{(2)}, \quad \Delta = p^2 + \mathsf{m}^2,\\ U^{(1)} &= -\gamma_\mu\hat{\mathsf{D}}_\mu\mathsf{m}, \quad U^{(2)} = -\frac{1}{2}\sigma_{\mu\nu}\mathsf{F}_{\mu\nu}.\end{aligned} \tag{36}$$

Before proceeding, two comments can be made. First, $\mathrm{DD}^\dagger$ can be used instead of $\mathrm{D}^\dagger\mathrm{D}$. It corresponds to the replacement $\mathsf{m}\to-\mathsf{m}$ in the previous expressions, so W^+ is an even functional of m. This is readily verified in the final expression. Second, D^2 would also produce another acceptable bosonic operator $P'^2 + U'$, with $P'_\mu = \mathsf{D}_\mu + \gamma_\mu\mathsf{m}$. However, this form is not suitable for a derivative expansion, because $\gamma_\mu\mathsf{m}$ is of zeroth order. This implies that low orders in the derivative expansion of the fermionic problem would pick up contributions from all orders in the bosonic derivative expansion.

Combining the previous equations, and after taking the Dirac trace, the following result is obtained[5]:

$$W_0^+[v,m] = -\frac{2^{d/2}}{2}\int\frac{\mathrm{d}^dx\mathrm{d}^dp}{(2\pi)^d}\mathrm{tr}\log\Delta, \tag{37}$$

$$\begin{aligned}W_2^+[v,m] &= -\frac{2^{d/2}}{2}\int\frac{\mathrm{d}^dx\mathrm{d}^dp}{(2\pi)^d}\\ &\quad\mathrm{tr}\left[-\frac{1}{2}\left(\frac{1}{\Delta}\mathsf{m}_\mu\right)^2 + \frac{p^2}{d}\left(\frac{1}{\Delta^2}(\mathsf{m}^2)_\mu\right)^2\right],\end{aligned} \tag{38}$$

$$\begin{aligned}W_4^+[v,m] &= -\frac{2^{d/2}}{2}\int\frac{\mathrm{d}^dx\mathrm{d}^dp}{(2\pi)^d}\\ &\quad\times\mathrm{tr}\left[-\frac{1}{2}\left(\frac{1}{\Delta}\mathsf{m}_\mu\right)^4 + \frac{1}{4}\left(\frac{1}{\Delta}\mathsf{m}_\mu\frac{1}{\Delta}\mathsf{m}_\nu\right)^2\right.\\ &\quad+\frac{1}{\Delta}\mathsf{m}_\mu\frac{1}{\Delta}\mathsf{m}_\nu\frac{1}{\Delta}\mathsf{F}_{\mu\nu} + \frac{1}{4}\left(\frac{1}{\Delta}\mathsf{F}_{\mu\nu}\right)^2\\ &\quad+\frac{p^2}{d}\left\{2\left(\frac{1}{\Delta^2}(\mathsf{m}^2)_\mu\frac{1}{\Delta}\mathsf{m}_\nu\right)^2 + 2\frac{1}{\Delta^2}(\mathsf{m}^2)_\mu\frac{1}{\Delta^2}\mathsf{m}_\nu\frac{1}{\Delta}\right.\\ &\quad\left.\times(\mathsf{m}^2)_\mu\frac{1}{\Delta}\mathsf{m}_\nu + 2\frac{1}{\Delta}(\mathsf{m}^2)_\mu\frac{1}{\Delta^2}(\mathsf{m}^2)_\mu\frac{1}{\Delta}\mathsf{m}_\nu\frac{1}{\Delta^2}\mathsf{m}_\nu\right.\end{aligned}$$

[4] A misprint in the formula presented in [8] was corrected in [9].

[5] We will use the notations $W_n^+[v,m]$ and $W_{n,d}^+[v,m]$ where n specifies the order in the derivative expansion and d the space-time dimension.

$$+4\left(\frac{1}{\Delta^2}(\mathsf{m}^2)_\mu\right)^2\left(\frac{1}{\Delta}\mathsf{m}_\nu\right)^2-4\frac{1}{\Delta^2}(\mathsf{m}^2)_\mu\frac{1}{\Delta}\mathsf{m}_\nu\frac{1}{\Delta^2}\mathsf{m}_{\mu\nu}$$
$$-4\frac{1}{\Delta^2}(\mathsf{m}^2)_\mu\frac{1}{\Delta^2}\mathsf{m}_\nu\frac{1}{\Delta}\mathsf{m}_{\mu\nu}+\left(\frac{1}{\Delta^2}\mathsf{m}_{\mu\nu}\right)^2\Bigg\}$$
$$-\frac{2p^4}{d(d+2)}\Bigg\{-2\left(\frac{1}{\Delta^2}(\mathsf{m}^2)_\mu\right)^4$$
$$+\left(\frac{1}{\Delta^2}(\mathsf{m}^2)_\mu\frac{1}{\Delta^2}(\mathsf{m}^2)_\nu\right)^2+8\left(\frac{1}{\Delta^3}(\mathsf{m}^2)_\mu\frac{1}{\Delta}(\mathsf{m}^2)_\mu\right)^2$$
$$+2\left(\frac{1}{\Delta^3}(\mathsf{m}^2)_{\mu\mu}\right)^2-8\frac{1}{\Delta^3}(\mathsf{m}^2)_\mu\frac{1}{\Delta}(\mathsf{m}^2)_\mu\frac{1}{\Delta^3}(\mathsf{m}^2)_{\nu\nu}$$
$$+4\frac{1}{\Delta^2}(\mathsf{m}^2)_\mu\frac{1}{\Delta^2}(\mathsf{m}^2)_\nu\frac{1}{\Delta^2}\mathsf{F}_{\mu\nu}+\left(\frac{1}{\Delta^2}\mathsf{F}_{\mu\nu}\right)^2\Bigg\}\Bigg]. \quad (39)$$

$2^{d/2}$ is the dimension of Dirac space and tr stands for trace on flavor space only. Two further notational conventions have been used. First, if I is an ordered set of Lorentz indices, $X_{\mu I}$ denotes[6] $\hat{\mathsf{D}}_\mu X_I$; thus, for instance,

$$\mathsf{m}_\mu=\hat{\mathsf{D}}_\mu\mathsf{m},\quad \mathsf{m}_{\mu\nu}=\hat{\mathsf{D}}_\mu(\hat{\mathsf{D}}_\nu\mathsf{m}). \quad (40)$$

Second, the bosonic determinant and all manipulations used are invariant under the transformation defined by

$$\mathsf{m}\mapsto\mathsf{m}^\dagger=\mathsf{m},\quad \mathsf{D}_\mu\mapsto\mathsf{D}_\mu^\dagger=-\mathsf{D}_\mu,\quad (XY)^\dagger=Y^\dagger X^\dagger \quad (41)$$

(this is just the Hermitian conjugation when the fields have the standard hermiticity). So, for instance,

$$\mathsf{m}_\mu^\dagger=\mathsf{m}_\mu,\quad \mathsf{m}_{\mu\nu}^\dagger=\mathsf{m}_{\mu\nu},\quad \mathsf{F}_{\mu\nu}^\dagger=-\mathsf{F}_{\mu\nu}. \quad (42)$$

Therefore, all terms in the previous formulas either are self-conjugated or come in conjugated pairs. In writing the formula for $W_4^+[v,m]$ each term has been identified with its conjugate [9,10]. For instance, $\Delta^{-2}(\mathsf{m}^2)_\mu\Delta^{-1}\mathsf{m}_\nu\Delta^{-2}\mathsf{m}_{\mu\nu}$, actually stands for

$$\frac{1}{2}\frac{1}{\Delta^2}(\mathsf{m}^2)_\mu\frac{1}{\Delta}\mathsf{m}_\nu\frac{1}{\Delta^2}\mathsf{m}_{\mu\nu}+\frac{1}{2}\frac{1}{\Delta^2}\mathsf{m}_\nu\frac{1}{\Delta}(\mathsf{m}^2)_\mu\frac{1}{\Delta^2}\mathsf{m}_{\mu\nu}. \quad (43)$$

(Recall that terms related by a cyclic permutation are identified due to the trace.)

In deriving the formulas for W^+ the identity

$$\left(\frac{1}{\Delta}\right)_\mu=-\frac{1}{\Delta}(\mathsf{m}^2)_\mu\frac{1}{\Delta} \quad (44)$$

has been employed. In general, it will not be practical to further expand $(\mathsf{m}^2)_\mu$ as $\mathsf{m}\mathsf{m}_\mu+\mathsf{m}_\mu\mathsf{m}$, since m^2 is a chirality preserving quantity and so it is better behaved than m. From the method used it follows that W^+ can be written in such a way that single factors m appear only through derivatives, i.e. m_μ, $\mathsf{m}_{\mu\nu}$, etc., and underivated m are always squared.

[6] Note that in [9] this is denoted by $X_{I\mu}$.

3.3 Momentum integrals

Using Convention 3, all momentum integrals in (37), (38) and (39) can explicitly be computed. The ultraviolet divergences can be dealt with by using dimensional regularization. The basic integral is of the form

$$I_{n,k,d}(z_1,\ldots,z_n;\epsilon)=\frac{(4\pi)^{\hat{d}/2}\Gamma(\hat{d}/2)}{(4\pi)^{d/2}\Gamma(d/2)}\int\frac{\mathrm{d}^{\hat{d}}p}{(2\pi)^{\hat{d}}}\left(p^2\right)^r\prod_{i=1}^n\frac{1}{\Delta_i}, \quad (45)$$

where

$$\hat{d}=d-2\epsilon,\quad k=\frac{d}{2}+r-1,\quad \Delta_i=p^2+z_i^2. \quad (46)$$

Here, $z_1,\ldots,z_n$ are independent non-vanishing variables (to be replaced later by m). The integral can be done straightforwardly and gives

$$I_{n,k,d}=\frac{\Gamma(k-\epsilon+1)\Gamma(-k+\epsilon)}{(4\pi)^{d/2}\Gamma(d/2)}\sum_{i=1}^n(z_i^2)^{k-\epsilon}R_{n,i}, \quad (47)$$

with

$$R_{n,i}(z_1,\ldots,z_n)=\prod_{j\neq i}\frac{1}{z_j^2-z_i^2}. \quad (48)$$

Finally, expanding in powers of ϵ yields

$$I_{n,k,d}=\frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\times\left(H_{n,k}+\left(\frac{1}{\epsilon}-\log(\mu^2)\right)Q_{n,k}+O(\epsilon)\right). \quad (49)$$

where the following functions have been introduced:

$$H_{n,k}(z_1,\ldots,z_n;\mu)=(-1)^{k+1}\sum_{i=1}^n(z_i^2)^k\log(z_i^2/\mu^2)R_{n,i},$$
$$Q_{n,k}(z_1,\ldots,z_n)=(-1)^k\sum_{i=1}^n(z_i^2)^kR_{n,i}. \quad (50)$$

Several comments are in order here.

(i) As indicated by the notation, the functions $H_{n,k}$ and $Q_{n,k}$ do not directly depend on the space-time dimension.

(ii) Following standard practice, the parameter μ^2 has been introduced for convenience in the dimensional counting. The integral $I_{n,k,d}$ does not depend on μ^2 and so its dependence cancels in the r.h.s. of (49). Under a scale transformation, $R_{n,i}$ and $Q_{n,k}$ transform homogeneously, but

$$z_i^2\mapsto\lambda z_i^2,\quad H_{n,k}\mapsto\lambda^{k+1-n}\left(H_{n,k}-\log(\lambda)Q_{n,k}\right). \quad (51)$$

Therefore, $Q_{n,k}$ represents the contribution of the integral to the scale anomaly.

(iii) Whenever the integral $I_{n,k,d}$ is infrared and ultraviolet convergent, the function $Q_{n,k}$ must vanish:

$$Q_{n,k} = 0 \quad \text{when} \quad 1 \leq k+1 < n. \tag{52}$$

This is because as ϵ goes to zero, $I_{n,k,d}$ has to remain finite. In addition, $\log(\mu^2)$ no longer appears and there is no scale anomaly. When the integral is infrared finite ($k \geq 0$, which is the case of interest), but not necessarily ultraviolet finite, the functions $Q_{n,k}$ are just homogeneous polynomials in the squared masses, z_i^2, since taking a sufficient number of derivatives of $I_{n,k,d}$ with respect to the any of the z_i^2, turns the integral into a ultraviolet convergent one.

(iv) The ultraviolet finite prefactor $((4\pi)^{\hat{d}/2}\Gamma(\hat{d}/2))/((4\pi)^{d/2}\Gamma(d/2))$ has been introduced in the definition of $I_{n,k,d}$ for convenience, in order to remove renormalization dependent constant terms (the usual $\log(4\pi)-\gamma$ of the MS scheme) in (49). Those constant terms would come with $Q_{n,k}$ and would amount to further polynomial contributions to the effective action.

(v) From the definition of $I_{n,k,d}$, it follows that $H_{n,k}$ and $Q_{n,k}$ are completely symmetric functions of the m_i^2, and furthermore, they remain finite as two or more of the squared masses become equal. ($Q_{n,k}$ is a polynomial so the finiteness is obvious in this case.)

(vi) The identity

$$\frac{1}{\Delta_i}\frac{1}{\Delta_j} = -\frac{1}{z_i^2 - z_j^2}\left(\frac{1}{\Delta_i} - \frac{1}{\Delta_j}\right) \tag{53}$$

gives rise to the following recurrence relation:

$$I_{n,k,d}(z_1, z_2, \ldots; \epsilon) = \tag{54}$$
$$\frac{I_{n-1,k,d}(z_2, z_3, \ldots, z_n; \epsilon) - I_{n-1,k,d}(z_1, z_3, \ldots, z_n; \epsilon)}{z_1^2 - z_2^2}.$$

As a consequence, analogous recurrence relations apply to $H_{n,k}$ and $Q_{n,k}$ as well, and they can be used to compute these functions starting from

$$H_{1,k} = (-1)^{k+1}(z_1^2)^k \log(z_1^2/\mu^2),$$
$$Q_{1,k} = (-1)^k (z_1^2)^k. \tag{55}$$

More generally, it will be necessary to consider also the case of Δ_i raised to different powers in the momentum integral, (45), namely

$$I^{r_1,\ldots,r_n}_{n,k,d}(z_1, \ldots, z_n; \epsilon)$$
$$= \frac{(4\pi)^{\hat{d}/2}\Gamma(\hat{d}/2)}{(4\pi)^{d/2}\Gamma(d/2)} \int \frac{\mathrm{d}^{\hat{d}}p}{(2\pi)^{\hat{d}}} \left(p^2\right)^r \prod_{i=1}^{n} \frac{1}{\Delta_i^{r_i}}. \tag{56}$$

The r_i are assumed to be positive integers. Obviously, this integral is given by the previous formulas by taking the first r_1 arguments to be all of them equal z_1, then the next r_2 arguments to be z_2, and so on. That is,

$$I^{r_1,\ldots,r_n}_{n,k,d}(z_1, \ldots, z_n; \epsilon)$$
$$= I_{r_1+\cdots+r_n,k,d}(z_1, \ldots, z_1, \ldots, z_n, \ldots, z_n; \epsilon), \tag{57}$$

where in the r.h.s. each z_i appears r_i times. The corresponding analogous definitions apply to $H^{r_1,\ldots,r_n}_{n,k}$ and $Q^{r_1,\ldots,r_n}_{n,k}$.

Alternatively, the same result is obtained taking derivatives with respect to the z_i^2. For instance,

$$H^{r_1,\ldots,r_n}_{n,k}(z_1, \ldots, z_n; \mu)$$
$$= \prod_{i=1}^{n}\left[\frac{(-1)^{r_i-1}}{(r_i-1)!}\frac{\partial^{r_i-1}}{\partial(z_i^2)^{r_i-1}}\right] H_{n,k}(z_1, \ldots, z_n; \mu). \tag{58}$$

Analogous formulas hold for $I^{r_1,\ldots,r_n}_{n,k,d}$ and $Q^{r_1,\ldots,r_n}_{n,k}$.

It is interesting to note that $H_{n,k}$ has the form of a rational function times a logarithm of z_i^2/μ^2 (see remark (iv) above). This needs not be true for $H^{r_1,\ldots,r_n}_{n,k}$; taking the coincidence limit $z_i = z_j$ or derivating with respect to z_i^2 yields further terms which are purely rational functions of the masses, without logarithms.

3.4 Normal parity effective action through fourth order

The momentum integrals in (37), (38) and (39) can straightforwardly be worked out using the previous formulas for $I^{r_1,\ldots,r_n}_{n,k,d}$. In the spirit of dimensional regularization, the variable d in (37), (38) and (39) (except that in $\mathrm{d}^d x$) should be replaced by $\hat{d} = d - 2\epsilon$ and a Laurent expansion in ϵ is to be performed. After that there will be two types of terms. First, contributions coming from $(1/\epsilon - \log(\mu^2))Q^{r_1,\ldots,r_n}_{n,k}$ in the momentum integral. These pick up ultraviolet divergent terms plus finite terms from the ϵ in $\hat{d}$. And second, there will be finite terms from $H^{r_1,\ldots,r_n}_{n,k}$. The first kind of contributions are just polynomials and so renormalization dependent. Because all our formulas are already manifestly gauge invariant before momentum integration, those polynomials are gauge invariant and removable by counterterms. (Within other schemes, the divergent polynomial terms would still be gauge invariant but not necessarily the finite parts.) Therefore, modulo polynomials, the effective action can be obtained by using d instead of $\hat{d}$ and keeping only the finite part $H^{r_1,\ldots,r_n}_{n,k}$ in the momentum integrals.

The calculation of $W_0^+[v,m]$ in (37) can be done by first taking the derivative with respect to m^2. This removes the logarithm and allows for the application of the momentum integrals derived in the previous subsection. $W_0^+[v,m]$ is then obtained by integrating back with respect to m^2. This introduces an ambiguity which is just a polynomial.

$$W^+_{0,d}[v,m] = -\frac{(-1)^{d/2}2^{d/2}}{2(4\pi)^{d/2}\Gamma(d/2+1)}$$
$$\times \int \mathrm{d}^d x \mathrm{tr}\left[\mathsf{m}^d \log(\mathsf{m}^2/\mu^2)\right]. \tag{59}$$

Using our Convention 3 and the integrals of the previous section, the expressions for $W_2^+[v,m]$ and $W_4^+[v,m]$

are a direct transcription of (38) and (39). In the following formulas the arguments $z_1, \ldots, z_n$ of $H^{r_1,\ldots,r_n}_{n,k}$ have to be substituted by $\mathsf{m}_1, \ldots, \mathsf{m}_n$.

$$W^+_{2,d}[v,m] = -\frac{1}{2}\frac{2^{d/2}}{(4\pi)^{d/2}\Gamma(d/2)} \times \int \mathrm{d}^d x \mathrm{tr}\left[-\frac{1}{2}H^{1,1}_{2,d/2-1}\mathsf{m}^2_\mu + \frac{1}{d}H^{2,2}_{2,d/2}(\mathsf{m}^2)^2_\mu\right], \quad (60)$$

$$\begin{aligned} W^+_{4,d}[v,m] &= -\frac{1}{2}\frac{2^{d/2}}{(4\pi)^{d/2}\Gamma(d/2)} \\ &\times \int \mathrm{d}^d x \mathrm{tr}\left[-\frac{1}{2}H^{1,1,1,1}_{4,d/2-1}\mathsf{m}^2_\mu\mathsf{m}^2_\nu + \frac{1}{4}H^{1,1,1,1}_{4,d/2-1}(\mathsf{m}_\mu\mathsf{m}_\nu)^2\right. \\ &+H^{1,1,1}_{3,d/2-1}\mathsf{m}_\mu\mathsf{m}_\nu\mathsf{F}_{\mu\nu} + \frac{1}{4}H^{1,1}_{2,d/2-1}\mathsf{F}^2_{\mu\nu} \\ &+\frac{2}{d}H^{2,1,2,1}_{4,d/2}((\mathsf{m}^2)_\mu\mathsf{m}_\nu)^2 + \frac{2}{d}H^{2,2,1,1}_{4,d/2}((\mathsf{m}^2)_\mu\mathsf{m}_\nu)^2 \\ &+\frac{2}{d}H^{1,2,1,2}_{4,d/2}(\mathsf{m}^2)^2_\mu\mathsf{m}^2_\nu + \frac{4}{d}H^{2,2,1,1}_{4,d/2}(\mathsf{m}^2)^2_\mu\mathsf{m}^2_\nu \\ &-\frac{4}{d}H^{2,1,2}_{3,d/2}(\mathsf{m}^2)_\mu\mathsf{m}_\nu\mathsf{m}_{\mu\nu} - \frac{4}{d}H^{2,2,1}_{3,d/2}(\mathsf{m}^2)_\mu\mathsf{m}_\nu\mathsf{m}_{\mu\nu} \\ &+\frac{1}{d}H^{2,2}_{2,d/2}\mathsf{m}^2_{\mu\nu} + \frac{4}{d(d+2)}H^{2,2,2,2}_{4,d/2+1}(\mathsf{m}^2)^2_\mu(\mathsf{m}^2)^2_\nu \\ &-\frac{2}{d(d+2)}H^{2,2,2,2}_{4,d/2+1}((\mathsf{m}^2)_\mu(\mathsf{m}^2)_\nu)^2 - \frac{16}{d(d+2)} \\ &\times H^{3,1,3,1}_{4,d/2+1}(\mathsf{m}^2)^2_\mu(\mathsf{m}^2)^2_\nu - \frac{4}{d(d+2)}H^{3,3}_{2,d/2+1}(\mathsf{m}^2)^2_{\mu\mu} \\ &+\frac{16}{d(d+2)}H^{3,1,3}_{3,d/2+1}(\mathsf{m}^2)^2_\mu(\mathsf{m}^2)_{\nu\nu} - \frac{8}{d(d+2)}H^{2,2,2}_{3,d/2+1} \\ &\left.\times(\mathsf{m}^2)_\mu(\mathsf{m}^2)_\nu\mathsf{F}_{\mu\nu} - \frac{2}{d(d+2)}H^{2,2}_{2,d/2+1}\mathsf{F}^2_{\mu\nu}\right]. \end{aligned} \quad (61)$$

4 Discussion

4.1 Expression in terms of matrix elements

In order to analyze the previous formulas, let us consider in more detail the second order term in two space-time dimensions, $W^+_{2,2}[v,m]$. A straightforward calculation yields

$$\begin{aligned} W^+_{2,2}[v,m] &= \frac{1}{8\pi}\int \mathrm{d}^2 x \mathrm{tr}\left[H(\mathsf{m}_1,\mathsf{m}_2)\mathsf{m}^2_\mu\right. \\ &\left. - \frac{(\mathsf{m}^2_1+\mathsf{m}^2_2)H(\mathsf{m}_1,\mathsf{m}_2)-2}{(\mathsf{m}^2_1-\mathsf{m}^2_2)^2}(\mathsf{m}^2)^2_\mu\right], \end{aligned} \quad (62)$$

where

$$H(x,y) = \frac{\log(x^2/y^2)}{x^2-y^2}. \quad (63)$$

The symmetry under the exchange of the ordering labels 1, 2 is a direct consequence of the cyclic property of the trace.

As noted, this kind of formulas can be evaluated using a basis of eigenvectors of m^2 at each point x. This gives (Convention 1 is still at work)

$$\begin{aligned} W^+_{2,2}[v,m] &= \frac{1}{8\pi}\int \mathrm{d}^2 x \sum_{j,k}\left[H(m_j,m_k)(\hat{D}_\mu m_{\mathrm{RL}})_{jk}(\hat{D}_\mu m_{\mathrm{LR}})_{kj}\right. \\ &\left. -\frac{(m^2_j+m^2_k)H(m_j,m_k)-2}{(m^2_j-m^2_k)^2}(\hat{D}_\mu m^2_{\mathrm{R}})_{jk}(\hat{D}_\mu m^2_{\mathrm{R}})_{kj}\right]. \end{aligned} \quad (64)$$

In this formula $(\hat{D}_\mu m_{\mathrm{RL}})_{jk} = \langle j,\mathrm{R}|\hat{D}_\mu m_{\mathrm{RL}}|k,\mathrm{L}\rangle$, etc. The quantities $m^2_{\mathrm{R,L}}$ were defined in (6).

Particular attention requires the case of m^2_1 and m^2_2 with the same eigenvalue. Before momentum integration it is clear that this case is a perfectly regular one. It follows that the correct result after momentum integration can be obtained by taking the finite formal limit $\mathsf{m}^2_2 \to \mathsf{m}^2_1$, and then replacing m^2 by its eigenvalue. That is, in (64) the case $j=k$ (or more generally, $m^2_j = m^2_k$) is resolved by taking the limit $m^2_k \to m^2_j$ which is well defined and finite, namely

$$\begin{aligned} &\frac{1}{8\pi}\int \mathrm{d}^2 x \sum_j\left(\frac{1}{m^2_j}(\hat{D}_\mu m_{\mathrm{RL}})_{jj}(\hat{D}_\mu m_{\mathrm{LR}})_{jj}\right. \\ &\left. -\frac{1}{6m^4_j}(\hat{D}_\mu m^2_{\mathrm{R}})^2_{jj}\right) \\ &= \frac{1}{8\pi}\int \mathrm{d}^2 x \sum_j\left(\frac{2}{3m^2_j}(\hat{D}_\mu m_{\mathrm{RL}})_{jj}(\hat{D}_\mu m_{\mathrm{LR}})_{jj}\right. \\ &\left. -\frac{1}{3m^2_j}(\hat{D}_\mu m_{\mathrm{RL}})^2_{jj}\right). \end{aligned} \quad (65)$$

This is the full result in the Abelian case.

As an application, consider $m_{\mathrm{LR}}(x) = M(x)U(x)$, $m_{\mathrm{RL}}(x) = M(x)U^{-1}(x)$, where $M(x)$ is a c-number but not necessarily constant. Since $\mathsf{m}^2 = M^2$ is a c-number, the calculation can be done directly from (62) by taking $\mathsf{m}^2 = M^2$ everywhere. This gives

$$W^+_{2,2} = \frac{1}{4\pi}\int \mathrm{d}^2 x \mathrm{tr}\left[\frac{1}{6}\left(\frac{\partial_\mu M}{M}\right)^2 - \frac{1}{2}(U^{-1}\hat{D}_\mu U)^2\right]. \quad (66)$$

4.2 Commutator expansions

The derivative expansion treats m non-perturbatively. In other approaches the result appears instead in the form of a $1/m$-like expansion such that each term is an homogeneous function of m [11,1]. The order of each term in such an expansion is counted by the number of commutators (including those implied by the covariant derivative). Therefore, when operators m in the middle of an expression are moved to the left, producing commutators, the new terms so generated count as higher order than the original one and hence they are suppressed in this counting. Although the present work is devoted to derivative expansions, in this subsection we will consider the commutator expansion of our formulas.

Using our Convention 3 it is remarkably simple to systematically reexpand W^+ in terms of commutators. This

can be illustrated with $W^+_{2,2}$ in the form given in (80) (this expression will be derived below). From (21) it follows that the quantity

$$\mathsf{c}_1 = \mathsf{m}_1 - \mathsf{m}_2 \tag{67}$$

is equivalent to a commutator $[\mathsf{m},\ ]$ on the first m_μ. For instance,

$$\begin{aligned}\mathsf{c}_1^2\mathsf{m}_\mu^2 &= (\mathsf{m}_1-\mathsf{m}_2)^2\mathsf{m}_\mu^2 = (\mathsf{m}_1-\mathsf{m}_2)[\mathsf{m},\mathsf{m}_\mu]\mathsf{m}_\mu \\ &= [\mathsf{m},[\mathsf{m},\mathsf{m}_\mu]]\mathsf{m}_\mu. \end{aligned}\tag{68}$$

Therefore, a commutator expansion is obtained by removing the variable m_2 in favor of c_1, i.e., $\mathsf{m}_2 = \mathsf{m}_1 - \mathsf{c}_1$, and carrying out an expansion in powers of c_1:

$$-\frac{\mathsf{m}_1\mathsf{m}_2 H(\mathsf{m}_1,\mathsf{m}_2)-1}{(\mathsf{m}_1-\mathsf{m}_2)^2} = \frac{1}{6}\frac{1}{\mathsf{m}_1^2} + \frac{1}{6}\frac{\mathsf{c}_1}{\mathsf{m}_1^3} + \frac{2}{15}\frac{\mathsf{c}_1^2}{\mathsf{m}_1^4} + \cdots \tag{69}$$

This immediately translates into

$$W^+_{2,2}[v,m] = \frac{1}{4\pi}\int \mathrm{d}^2x \mathrm{tr}\left[\frac{1}{6\mathsf{m}^2}\mathsf{m}_\mu^2 + \frac{1}{6\mathsf{m}^3}[\mathsf{m},\mathsf{m}_\mu]\mathsf{m}_\mu + \frac{2}{15\mathsf{m}^3}[\mathsf{m},[\mathsf{m},\mathsf{m}_\mu]]\mathsf{m}_\mu + \cdots\right]. \tag{70}$$

For W^+_4 (or higher orders) the identity $\mathsf{c}_n = \mathsf{m}_n - \mathsf{m}_{n+1}$ can be used, where c_n denotes the commutator $[\mathsf{m},\ ]$ on the n-th fixed element.

Due to the cyclic property (and in particular integration by parts when D_μ is involved) neither the derivative expansion nor the commutator expansion take a unique form. However, there is an important difference between both expansions regarding their uniqueness, namely, the derivative expansion is related to (gauge covariant) dilatations of the external fields v and m; each order is tied to a given power of the dilatation parameter. This guarantees that each order in the derivative expansion is a well-defined functional of v and m. (That is, each given order can be written in different forms but has a single numerical value for each given configuration of the fields.) On the other hand, the commutator expansion is not tied to any expansion parameter and so it is possible to have two different commutators expansions of a single functional which differ numerically at every order. For instance, in the identity

$$\mathrm{tr}(2\mathsf{m}[\mathsf{m},\mathsf{m}_\mu]\mathsf{m}_\mu) = \mathrm{tr}([\mathsf{m},[\mathsf{m},\mathsf{m}_\mu]]\mathsf{m}_\mu) \tag{71}$$

the same functional is represented by two terms of different order (i.e., with a different number of commutators), even imposing a standard form (m_1 and c_1 have been chosen as the independent variables). More generally, a functional of the form $\mathrm{tr}(f(\mathsf{m}_1,\mathsf{m}_2)\mathsf{m}_\mu^2)$ depends only on the symmetric component of the function $f(x_1,x_2)$, but the antisymmetric component may have a non-vanishing commutator expansion when m_1 and c_1 are used as the independent variables. This suggest to first symmetrize f and then expand[7]. In any case, the usefulness for our notation to analyze this kind of problems and even more for more complicated cases, such as contributions to W^+_4 or higher orders, is clear.

The ambiguity in the commutator expansion allows one to reorder $W^+_{2,2}$ so that the symmetry under the exchange of the ordering labels 1, 2 in the right-hand side of (69) is restored. A convenient choice, containing even orders only, is

$$-\frac{\mathsf{m}_1\mathsf{m}_2 H(\mathsf{m}_1,\mathsf{m}_2)-1}{(\mathsf{m}_1-\mathsf{m}_2)^2} = \frac{1}{6}\frac{1}{\mathsf{m}_1\mathsf{m}_2} + \frac{1}{30}\frac{\mathsf{c}_1}{\mathsf{m}_1^2}\frac{\mathsf{c}_2}{\mathsf{m}_2^2} + \frac{1}{140}\frac{\mathsf{c}_1^2}{\mathsf{m}_1^3}\frac{\mathsf{c}_2^2}{\mathsf{m}_2^3} + \cdots, \tag{72}$$

where $\mathsf{c}_2 = \mathsf{m}_2 - \mathsf{m}_3 = -\mathsf{c}_1$ represents $[\mathsf{m},\ ]$ on the second m_μ. This expansion can be obtained from that in (69) by recursively subtracting the symmetric version of the leading term and then symmetrizing the remainder. Equation (72) translates into

$$\begin{aligned}&W^+_{2,2}[v,m] \\ &= \frac{1}{4\pi}\int \mathrm{d}^2x \mathrm{tr}\left[\frac{1}{6}\left(\frac{1}{\mathsf{m}}\mathsf{m}_\mu\right)^2 + \frac{1}{30}\left(\frac{1}{\mathsf{m}^2}[\mathsf{m},\mathsf{m}_\mu]\right)^2 + \frac{1}{140}\left(\frac{1}{\mathsf{m}^3}[\mathsf{m},[\mathsf{m},\mathsf{m}_\mu]]\right)^2 + \cdots\right].\end{aligned}\tag{73}$$

The commutator expansion just derived is well suited for the vector-like case. In many cases, however, an expansion involving commutators of m^2 will be more useful. For instance, in the Abelian case m^2 is always a (not necessarily constant) c-number whereas m is not (i.e. $[\mathsf{m},\mathsf{X}]$ needs not vanish even when X is a multiplicative operator). The method to obtain such an expansion is the same as above and can be applied to $W^+_{2,2}$ in (62). Namely, the quantity

$$\mathsf{C}_1 = \mathsf{m}_1^2 - \mathsf{m}_2^2 \tag{74}$$

represents $[\mathsf{m}^2,\ ]$ applied to the first fixed factor (m_μ or $(\mathsf{m}^2)_\mu$) thus m_2^2 can eliminated in favor of C_1 in (62). This gives

$$\begin{aligned}&H(\mathsf{m}_1,\mathsf{m}_2) \\ &\quad= \frac{1}{\mathsf{m}_1^2} + \frac{\mathsf{C}_1}{2\mathsf{m}_1^4} + \frac{\mathsf{C}_1^2}{3\mathsf{m}_1^6} + \frac{\mathsf{C}_1^3}{4\mathsf{m}_1^8} + \frac{\mathsf{C}_1^4}{5\mathsf{m}_1^{10}} + \cdots, \\ &\frac{(\mathsf{m}_1^2+\mathsf{m}_2^2)H(\mathsf{m}_1,\mathsf{m}_2)-2}{(\mathsf{m}_1^2-\mathsf{m}_2^2)^2} \\ &\quad= \frac{1}{6\mathsf{m}_1^4} + \frac{\mathsf{C}_1}{6\mathsf{m}_1^6} + \frac{3\mathsf{C}_1^2}{20\mathsf{m}_1^8} + \frac{2\mathsf{C}_1^3}{15\mathsf{m}_1^{10}} + \frac{5\mathsf{C}_1^4}{42\mathsf{m}_1^{12}} + \cdots.\end{aligned}\tag{75}$$

Further, applying a (in this case partial) symmetrization in the labels 1, 2 to remove terms with an odd number of commutators yields

[7] What seems to be true is that, for a given non-vanishing functional, the order of the leading term (i.e., the number of commutators in the term with the least number of commutators) has an upper bound over the set of all possible commutators expansions of the functional.

$$
\begin{aligned}
&H(\mathsf{m}_1,\mathsf{m}_2)\\
&\quad=\frac{1}{\mathsf{m}_1^2}\left(1+\frac{1}{6}\frac{\mathsf{C}_1\mathsf{C}_2}{\mathsf{m}_1^2\mathsf{m}_2^2}+\frac{1}{30}\frac{\mathsf{C}_1^2\mathsf{C}_2^2}{\mathsf{m}_1^4\mathsf{m}_2^4}+\cdots\right),\\
&\frac{(\mathsf{m}_1^2+\mathsf{m}_2^2)H(\mathsf{m}_1,\mathsf{m}_2)-2}{(\mathsf{m}_1^2-\mathsf{m}_2^2)^2}\\
&\quad=\frac{1}{6}\frac{1}{\mathsf{m}_1^2\mathsf{m}_2^2}+\frac{1}{60}\frac{\mathsf{C}_1\mathsf{C}_2}{\mathsf{m}_1^4\mathsf{m}_2^4}+\frac{1}{420}\frac{\mathsf{C}_1^2\mathsf{C}_2^2}{\mathsf{m}_1^6\mathsf{m}_2^6}+\cdots. \qquad (76)
\end{aligned}
$$

(Here $\mathsf{C}_2=\mathsf{m}_2^2-\mathsf{m}_3^2=-\mathsf{C}_1$.) This translates into

$$
\begin{aligned}
&W^+_{2,2}[v,m]\\
&\quad=\frac{1}{8\pi}\int \mathrm{d}^2x\mathrm{tr}\left[\frac{1}{\mathsf{m}^2}\left(\mathsf{m}_\mu^2+\frac{1}{6}\left(\frac{1}{\mathsf{m}^2}[\mathsf{m}^2,\mathsf{m}_\mu]\right)^2\right.\right.\\
&\quad+\left.\frac{1}{30}\left(\frac{1}{\mathsf{m}^4}[\mathsf{m}^2,[\mathsf{m}^2,\mathsf{m}_\mu]]\right)^2+\cdots\right)\\
&\quad-\frac{1}{6}\left(\frac{1}{\mathsf{m}^2}(\mathsf{m}^2)_\mu\right)^2-\frac{1}{60}\left(\frac{1}{\mathsf{m}^4}[\mathsf{m}^2,(\mathsf{m}^2)_\mu]\right)^2\\
&\quad-\left.\frac{1}{420}\left(\frac{1}{\mathsf{m}^6}[\mathsf{m}^2,[\mathsf{m}^2,(\mathsf{m}^2)_\mu]]\right)^2+\cdots\right]. \qquad (77)
\end{aligned}
$$

In fully expanded notation the leading term is (compare with (65))

$$
\begin{aligned}
&W^+_{2,2,\mathrm{leading}}[v,m]\\
&=\frac{1}{16\pi}\int \mathrm{d}^2x\mathrm{tr}\left[\frac{1}{m_\mathrm{R}^2}\hat{D}_\mu m_\mathrm{RL}\hat{D}_\mu m_\mathrm{LR}+\frac{1}{m_\mathrm{L}^2}\hat{D}_\mu m_\mathrm{LR}\right.\\
&\left.\times\hat{D}_\mu m_\mathrm{RL}-\frac{1}{6}\left(\frac{1}{m_\mathrm{R}^2}\hat{D}_\mu m_\mathrm{R}^2\right)^2-\frac{1}{6}\left(\frac{1}{m_\mathrm{L}^2}\hat{D}_\mu m_\mathrm{L}^2\right)^2\right]. \qquad (78)
\end{aligned}
$$

In the example considered above, $\mathsf{m}^2=M^2$ a (not necessarily constant) c-number, this is the full contribution.

4.3 Alternative expressions

The expression in (62) can be somewhat simplified by using the identity (valid inside the trace)

$$
\begin{aligned}
(\mathsf{m}^2)_\mu^2=\{\mathsf{m},\mathsf{m}_\mu\}^2&=(\mathsf{m}_1+\mathsf{m}_2)(\mathsf{m}_2+\mathsf{m}_3)\mathsf{m}_\mu^2\\
&=(\mathsf{m}_1+\mathsf{m}_2)^2\mathsf{m}_\mu^2 \qquad (79)
\end{aligned}
$$

(recall that $\mathsf{m}_3=\mathsf{m}_1$ due to the cyclic property). This yields the formula

$$
W^+_{2,2}[v,m]=-\frac{1}{4\pi}\int \mathrm{d}^2x\mathrm{tr}\left[\frac{\mathsf{m}_1\mathsf{m}_2H(\mathsf{m}_1,\mathsf{m}_2)-1}{(\mathsf{m}_1-\mathsf{m}_2)^2}\mathsf{m}_\mu^2\right]. \qquad (80)
$$

In most instances, in order to use this kind of formulas it should be clear which are the chiral labels L, R in each of the factors. The simplest way to determine this is by forcing that the ordering labels 1, 2 appear only in m^2, since this object is either RR or LL and so does not flip the chirality label. This can always be achieved by splitting the expression into components which are even or odd under $\mathsf{m}_1\to-\mathsf{m}_1$ and $\mathsf{m}_2\to-\mathsf{m}_2$, that is, in an expression of the form $\mathrm{tr}[f(\mathsf{m}_1,\mathsf{m}_2)\mathsf{m}_\mu^2]$

$$
f(\mathsf{m}_1,\mathsf{m}_2)=A(\mathsf{m}_1^2,\mathsf{m}_2^2)+\mathsf{m}_1\mathsf{m}_2B(\mathsf{m}_1^2,\mathsf{m}_2^2). \qquad (81)
$$

(Note that any expression must be even under $\mathsf{m}\to-\mathsf{m}$ since there should be the same number of R and L labels. Therefore, no terms of the form $\mathsf{m}_1D(\mathsf{m}_1^2,\mathsf{m}_2^2)+\mathsf{m}_2F(\mathsf{m}_1^2,\mathsf{m}_2^2)$ can appear.) Applying this procedure to (80) yields

$$
\begin{aligned}
&W^+_{2,2}[v,m]\\
&=-\frac{1}{4\pi}\int \mathrm{d}^2x\mathrm{tr}\left[\frac{(\mathsf{m}_1^2+\mathsf{m}_2^2)H(\mathsf{m}_1,\mathsf{m}_2)-2}{(\mathsf{m}_1^2-\mathsf{m}_2^2)^2}(\mathsf{m}\mathsf{m}_\mu)^2\right.\\
&\left.+\frac{2\mathsf{m}_1^2\mathsf{m}_2^2H(\mathsf{m}_1,\mathsf{m}_2)-(\mathsf{m}_1^2+\mathsf{m}_2^2)}{(\mathsf{m}_1^2-\mathsf{m}_2^2)^2}\mathsf{m}_\mu^2\right]. \qquad (82)
\end{aligned}
$$

Now, from our conventions it unambiguously follows that in the first term the chiral labels of m_1^2 and m_2^2 as well as the labels of both factors $(\mathsf{m}\mathsf{m}_\mu)$ are RR. In the second term, m_1^2 is RR, m_2^2 is LL, the first factor m_μ is RL and the second one is LR. (Recall that under Convention 1, each term is identified with its pseudo-parity conjugate.) This allows to take matrix elements as in (64) in a straightforward manner. (The original expression (62) was already in this form.) The procedure expressed by (81) shows that any expression, such as that in (80), can be brought to a standard form, (82), which is free of ambiguities, i.e., the expression can be expanded undoing the Conventions 1 and 2 in an unambiguous way. This procedure immediately extends to more than two variables, as required in W_4^+ or higher order terms.

An essential point in this discussion is that, in an expression such as $\mathrm{tr}(f(\mathsf{m}_1,\mathsf{m}_2)\mathsf{m}_\mu^2)$, the function $f(x_1,x_2)$ must be regular in the coincidence limits. To further analyze this point, consider the contributions to $W^+_{2,2}$ in (80) coming solely from v_μ, i.e. setting ∂_μ to zero (recall that $\mathsf{D}_\mu=\partial_\mu+\mathsf{v}_\mu$). Using the identity

$$
[\mathsf{v}_\mu,\mathsf{m}]^2=-(\mathsf{m}_1-\mathsf{m}_2)^2\mathsf{v}_\mu^2, \qquad (83)
$$

which holds inside the trace, this gives

$$
\begin{aligned}
W^+_{2,2}[v,m]=\frac{1}{4\pi}\int \mathrm{d}^2x\mathrm{tr}\left[(\mathsf{m}_1\mathsf{m}_2H(\mathsf{m}_1,\mathsf{m}_2)-1)\,\mathsf{v}_\mu^2\right]\\
+O(\partial). \qquad (84)
\end{aligned}
$$

Because the action depends exclusively on the combination $\mathsf{D}=\partial+\mathsf{v}$ and no algebraic assumption has been made on v (it can be an arbitrary matrix), it is clear that the whole effective action can be reconstructed from the case $\partial_\mu=0$ by means of the replacement $\mathsf{v}\to\mathsf{D}$ everywhere (this a reciprocal of the usual gauging procedure). So in a formal sense

$$
W^+_{2,2}[v,m]=\frac{1}{4\pi}\int \mathrm{d}^2x\mathrm{tr}\left[(\mathsf{m}_1\mathsf{m}_2H(\mathsf{m}_1,\mathsf{m}_2)-1)\,\mathsf{D}_\mu^2\right]. \qquad (85)
$$

This is meaningful provided that D_μ (or v_μ prior to the "gauging") appears only in commutators. Under our Convention 3, this can be easily enforced through the formal identity

$$\mathsf{D}_\mu^2 = -\frac{\mathsf{m}_\mu^2}{(\mathsf{m}_1-\mathsf{m}_2)^2}. \tag{86}$$

The minus sign comes because $\mathsf{m}_\mu = \mathsf{D}_\mu\mathsf{m} - \mathsf{m}\mathsf{D}_\mu = (\mathsf{m}_2-\mathsf{m}_1)\mathsf{D}_\mu$ for the first D_μ in (85), whereas $\mathsf{m}_\mu = (\mathsf{m}_3-\mathsf{m}_2)\mathsf{D}_\mu = (\mathsf{m}_1-\mathsf{m}_2)\mathsf{D}_\mu$ for the second one. This immediately recovers the correct expression (80).

This does not mean however that the formal procedure is always justified. In fact, we could have considered, instead of (86), a different formal expression, namely

$$\mathsf{D}_\mu^2 = -\frac{(\mathsf{m}^2)_\mu^2}{(\mathsf{m}_1^2-\mathsf{m}_2^2)^2}. \tag{87}$$

This can be used in (85) and the result can be rewritten (in the spirit of (81)) as

$$W_{2,2}^+[v,m] = -\frac{1}{4\pi}\int \mathrm{d}^2x \mathrm{tr}\frac{1}{(\mathsf{m}_1^2-\mathsf{m}_2^2)^2} \times\left[H(\mathsf{m}_1,\mathsf{m}_2)(\mathsf{m}(\mathsf{m}^2)_\mu)^2 - (\mathsf{m}^2)_\mu^2\right]. \tag{88}$$

This expression is incorrect if taken naively. Because it involves derivatives of m^2 only, it would predict, for instance, a vanishing value for $W_{2,2}^+$ when m is on the chiral circle, i.e. when m^2 is a constant c-number, or that $W_{2,2}^+$ does not depend on the axial field in the Abelian case, both predictions being wrong. The reason is that (86) introduces a singularity at $\mathsf{m}_1=\mathsf{m}_2$ which is canceled by the numerator $\mathsf{m}_1\mathsf{m}_2H(\mathsf{m}_1,\mathsf{m}_2)-1$, but (87) introduces a new singularity at $\mathsf{m}_1=-\mathsf{m}_2$ which is not canceled; each of the terms in (88) is not separately finite when $\mathsf{m}_1^2=\mathsf{m}_2^2$ and in general they cannot cancel to each other due to their different chiral labels. This renders the expression meaningless unless $(\mathsf{m}^2)_\mu$ is expanded to cancel the spurious singularity at $\mathsf{m}_1=-\mathsf{m}_2$.

The fact that the formal expression (85) can be promoted to a regular expression with D_μ in commutators (cf. (80)) is a manifestation of gauge invariance. It is perhaps interesting to note that the gauge invariance of (85) can be checked even without actually bringing D_μ into commutators. Namely, D_μ appears only in commutators in an expression, if and only if, the expression remains invariant under the shift $\mathsf{D}_\mu \to \mathsf{D}_\mu + a_\mu$, where a_μ is an arbitrary constant c-number. That the formula (85) is gauge invariant can be seen by applying this test: it is readily seen that in the terms with a_μ, m_1 and m_2 coincide (since m and a_μ commute) and the limit $\mathsf{m}_1\to\mathsf{m}_2$ gives zero. (That it is a double zero follows from symmetry under exchange of the labels 1, 2.)

Finally, let us mention that (84), and thus (85), can be obtained rather directly in several ways without using Chan's method. The key point is that in the absence of ∂_μ all quantities are multiplicative operators and the cyclic property holds. The procedure implied by combining (85) and (86) can then be extended to compute $W_2^+[v,m]$ in any number of dimensions. However, this method is not suitable to treat the case of four or more derivatives. For instance, the formula similar to (85) to fourth order is

$$\begin{aligned} W_{4,2}^+ = \frac{1}{4\pi}\int \mathrm{d}^2x \mathrm{tr}\,[&((\mathsf{m}_1\mathsf{m}_3-\mathsf{m}_2\mathsf{m}_4)H_{4,1} \\ &+ \mathsf{m}_1\mathsf{m}_2\mathsf{m}_3\mathsf{m}_4H_{4,0})\,\mathsf{D}_\mu\mathsf{D}_\mu\mathsf{D}_\nu\mathsf{D}_\nu \\ &- \frac{1}{2}((\mathsf{m}_1\mathsf{m}_3+\mathsf{m}_2\mathsf{m}_4)H_{4,1} \\ &+ \mathsf{m}_1\mathsf{m}_2\mathsf{m}_3\mathsf{m}_4H_{4,0})\,\mathsf{D}_\mu\mathsf{D}_\nu\mathsf{D}_\mu\mathsf{D}_\nu], \end{aligned} \tag{89}$$

where $H_{4,k}$ are the functions of $\mathsf{m}_1,\mathsf{m}_2,\mathsf{m}_3,\mathsf{m}_4$ defined in (50). Using their explicit form and the test noted above of shifting D_μ by a constant c-number, it is possible to check that the previous formula is chiral invariant (of course, this is just a check of the calculation). However, there is no simple systematic procedure for bringing it to an explicitly invariant form where all covariant derivatives appear only in commutators. The trick introduced in (86) works in W_2^+, because D_μ appears there as a first covariant derivative. Such a replacement is no longer sufficient in W_4^+ since explicit chiral invariance requires at least the presence of second covariant derivatives as well as $\mathsf{F}_{\mu\nu}$. (If only the replacement in (86) is used, one obtains an expression involving m_μ with coefficient which are functions of $\mathsf{m}_1,\mathsf{m}_2,\mathsf{m}_3,\mathsf{m}_4$. However, these functions are not finite in the coincidence limit, i.e. when two or more of their arguments become equal, and such an expression is not truly well defined.)

4.4 Bosonic formulas

For completeness we give here the analogous formulas of (37), (38) and (39), and (59), (60) and (61) for the bosonic effective action:

$$\begin{aligned} \mathrm{Tr}\log(P^2+U) = \int\frac{\mathrm{d}^dx\mathrm{d}^dp}{(2\pi)^d}\mathrm{tr}\Bigg[&\log\Delta + \frac{p^2}{d}\left(\frac{1}{\Delta^2}U_\mu\right)^2 \\ &-\frac{2p^4}{d(d+2)}\Bigg\{-2\left(\frac{1}{\Delta^2}U_\mu\right)^4 + \left(\frac{1}{\Delta^2}U_\mu\frac{1}{\Delta^2}U_\nu\right)^2 \\ &+8\left(\frac{1}{\Delta^3}U_\mu\frac{1}{\Delta}U_\mu\right)^2 + 2\left(\frac{1}{\Delta^3}U_{\mu\mu}\right)^2 \\ &-8\frac{1}{\Delta^3}U_\mu\frac{1}{\Delta}U_\mu\frac{1}{\Delta^3}U_{\nu\nu} + 4\frac{1}{\Delta^2}U_\mu\frac{1}{\Delta^2}U_\nu\frac{1}{\Delta^2}F_{\mu\nu} \\ &+\left(\frac{1}{\Delta^2}F_{\mu\nu}\right)^2\Bigg\} + \cdots\Bigg], \end{aligned} \tag{90}$$

where $\Delta = p^2+U$, $U_\mu = [P_\mu, U]$, etc. From this formula, the fermionic one is obtained by setting $U = \mathsf{m}^2 + U^{(1)} + U^{(2)}$.

After momentum integration, the renormalized bosonic effective action becomes

$$\begin{aligned} &\mathrm{Tr}\log(P^2+U) \\ &= \frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\int \mathrm{d}^dx\mathrm{tr}\left[(-1)^{d/2}\frac{2}{d}U^{d/2}\log(U/\mu^2)\right. \end{aligned}$$

$$+\frac{1}{d}H^{2,2}_{2,d/2}U^2_\mu+\frac{4}{d(d+2)}H^{2,2,2,2}_{4,d/2+1}U^2_\mu U^2_\nu-\frac{2}{d(d+2)}$$
$$\times H^{2,2,2,2}_{4,d/2+1}(U_\mu U_\nu)^2-\frac{16}{d(d+2)}H^{3,1,3,1}_{4,d/2+1}U^2_\mu U^2_\nu$$
$$-\frac{4}{d(d+2)}H^{3,3}_{2,d/2+1}U^2_{\mu\mu}+\frac{16}{d(d+2)}H^{3,1,3}_{3,d/2+1}U^2_\mu U_{\nu\nu}$$
$$-\frac{8}{d(d+2)}H^{2,2,2}_{3,d/2+1}U_\mu U_\nu F_{\mu\nu}$$
$$-\frac{2}{d(d+2)}H^{2,2}_{2,d/2+1}F^2_{\mu\nu}+\cdots\Big]. \qquad (91)$$

In this case, the arguments $z_1^2,\ldots,z_n^2$ in $H^{r_1,\ldots,r_n}_{n,k}$ have to be substituted for by $U_1,\ldots,U_n$.

Acknowledgements. This work was supported in part by funds provided by the Spanish DGICYT grant no. PB98-1367 and Junta de Andalucía grant no. FQM-225.

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